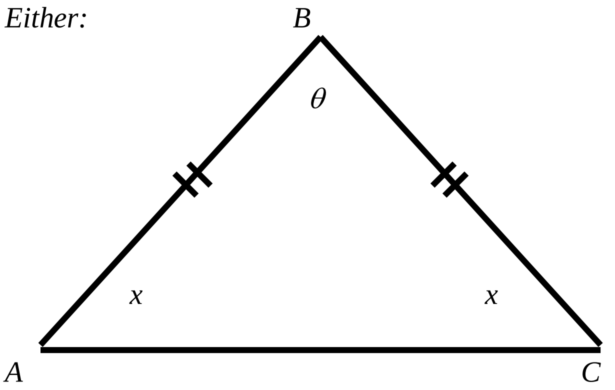
Geometric Proof Test – Solutions

Total = 28 marks. Suggested time = 60 minutes.

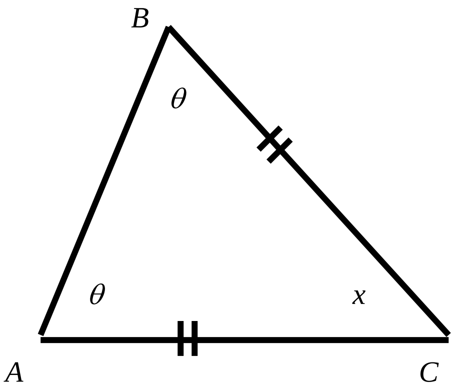
1. [4 marks] Triangle *ABC* is isosceles. Given that the size of $\angle ABC$ is $\theta°$

(a) Sketch the two distinct situations that can arise for triangle *ABC*

Either:



Or:



Diagrams ✓✓

(b) Prove that the size of $\angle BCA$ for one situation is $\frac{180-\theta}{2}$ and for the other situation the size of $\angle BCA$ is $180 - 2\theta$

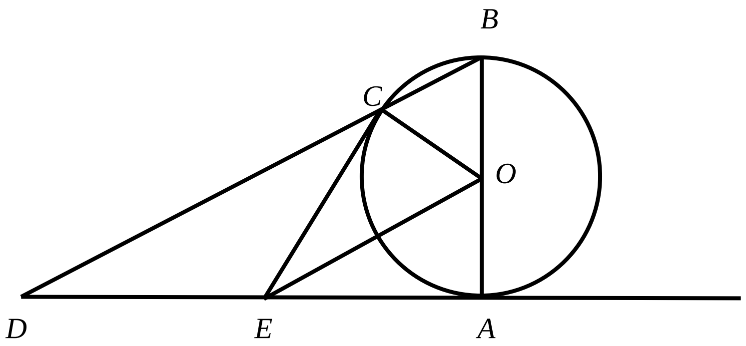
Let the size of $\angle BCA$ be x.

$2x + \theta = 180$ $2\theta - x = 180$

$x = \frac{180-\theta}{2}$ $x = 180 - 2\theta$

✓✓

2. [7 marks]



The diagram shows a circle centred at *O* with diameter $\overline{AB}$ and tangents $\overline{EC}$ and $\overline{DA}$.

Complete the seven blank Explanation cells in the tables below.

(a) Required to prove: triangle OAE is congruent to triangle OCE.

Statement	**Explanation**
$\overline{OA} \cong \overline{OC}$	Both are radii.
$\angle OAE \cong \angle OCE$	Both are 90°, the angle between a radius and a tangent.
(Note that ΔOAE and ΔOCE share a common side.)	
$\Delta OAE \cong \Delta OCE$	RHS

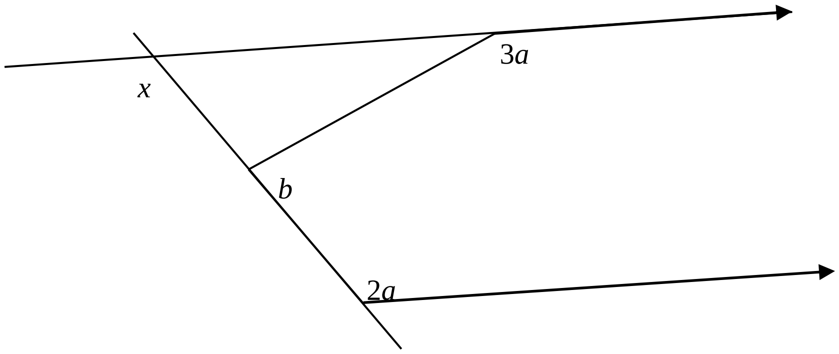
✓✓

(b) Required to prove: E is the midpoint of $\overline{DA}$. That is, $\overline{AE} : \overline{AD} = 1 : 2$

Statement	**Explanation**
$\angle AOE \cong \angle COE$	Corresponding parts of congruent shapes, as shown in the table above.
Let the size of $\angle AOE$ and $\angle COE$ be θ.	
$\angle COB = 180 - 2\theta$	Angles on a straight line are supplementary.
$\angle OCB = \angle OBC = \theta$	ΔOCB is isosceles, and $\angle COB = 180 - 2\theta$
$\overline{OE} \,//\, \overline{BC}$	$\angle$COE and $\angle$OCB are alternate angles $\therefore$ the lines are parallel.
$\Delta OAE \sim \Delta BAD$	AAA
$\overline{OA} : \overline{AB} = 1 : 2$	Radius : Diameter
$\overline{AE} : \overline{AD} = 1 : 2$	Corresponding parts of similar shapes.

✓✓✓✓✓

3. [4 marks] *Diagram is not drawn to scale.*



Required to prove: $5a + b = 360°$ (*May not require the use of every row given in the table.*)

Addition of an appropriate extra line. ✓

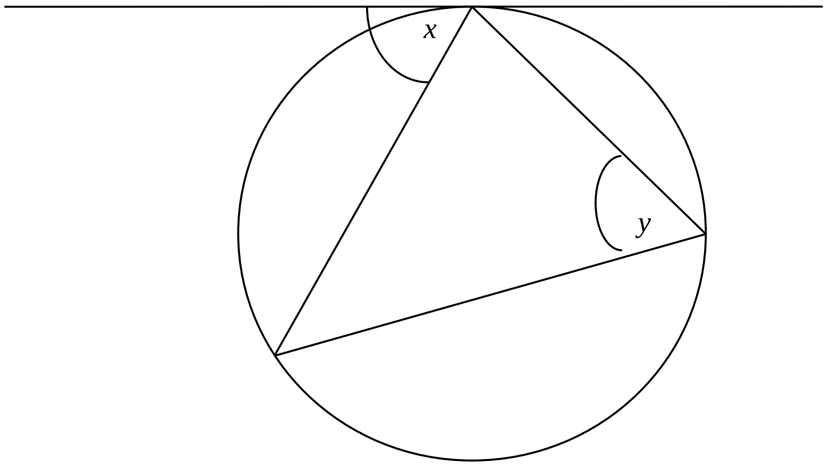
Statement	**Explanation**
$x = 2a$	Alternate angles on parallel lines are equal.
$x + 3a + b = 360$	Exterior angles of a polygon total 360°.
$5a = b = 360$	As required.

✓✓✓

Alternative proofs are possible, for example adding a transversal to form a pentagon, relying on the fact that the sum of the interior angles of a pentagon is 540°.

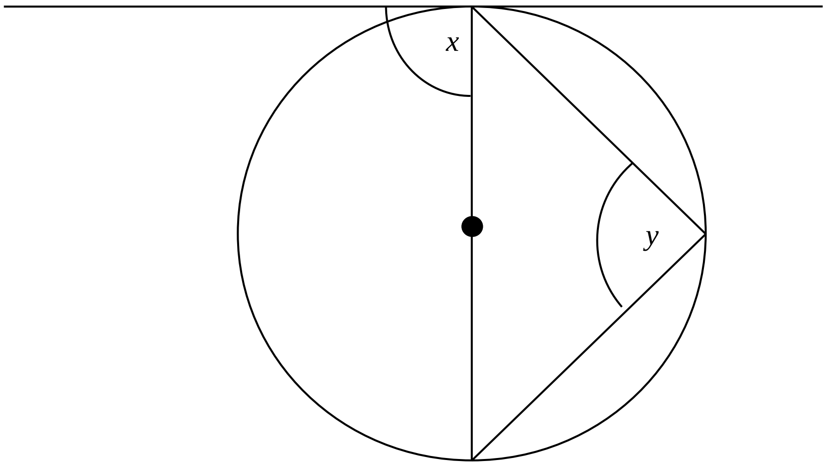
4. [5 marks]

(a) The Alternate Segment Theorem says that for a triangle with a circumcircle and a tangent at one vertex as drawn below, $\angle x = \angle y$. The letters are the sizes of the angles, in degrees.



A student, required to prove the Alternate Segment Theorem, gave this "proof":

Since the AST has to be true for all triangles, it has to be true when one side of the triangle is the diameter of the circle.



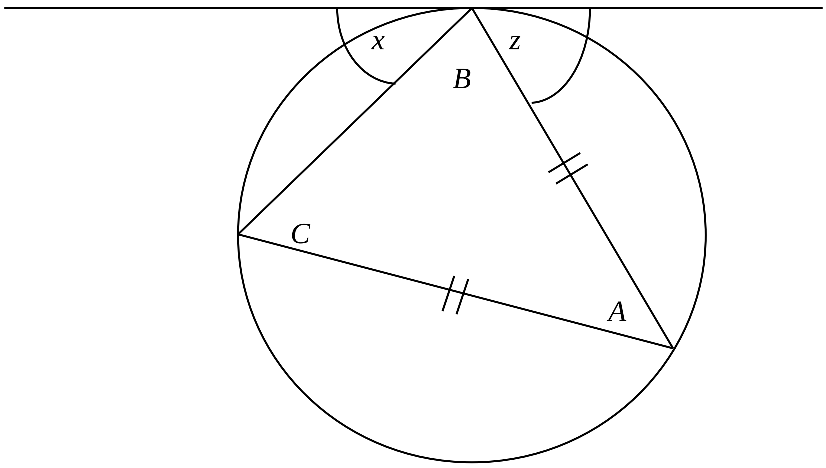
$\angle x = 90°$ because it is the angle between a radius and a tangent. $\angle y = 90°$ because it is the angle from a diameter subtended at the circumference. Therefore, $\angle x = \angle y$.

Explain anything that is wrong with the student's "proof".

A general rule cannot be proven by a particular case. ✓✓

(b) This diagram shows an isosceles triangle with its circumcircle and a tangent at one vertex.

The letters are the sizes of the angles, in degrees. Prove that $z = 90 - \frac{x}{2}$



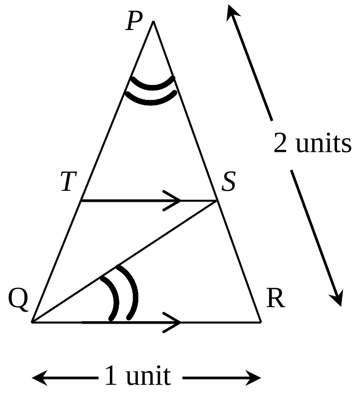
(*May not require the use of every row given in the table. Algebra may be shown below the table.*)

Statement	**Explanation**
$A \cong x, \quad C \cong z.$	Alternate Segment Theorem.
$B \cong C$	They are the equal angles in an isosceles triangle.
$A + B + C = 180°$	The sum of the interior angles of a triangle is 180°.
$x + z + z = 180°$	Substitutions as established above.
$z = \frac{180 - z}{2} = 90 - \frac{x}{2}$	As required.

✓✓✓

5. [4 marks] In the diagram to the right the length $\overline{PR}$ is twice length $\overline{QR}$, ∠QPR ≅ ∠SQR and the lines $\overline{TS}$ and $\overline{QR}$ are parallel.

Diagram is not drawn to scale.



(a) Prove that $\overline{PR}$ is 4 times the length of $\overline{SR}$.

Statement	Explanation
$\Delta PQR \sim \Delta QRS$	AAA (∠*PRQ* is common.)
$\overline{QR} : \overline{PR} = 1:2$	Given.
$\overline{SR} : \overline{QR} = 1:2$	Corresponding parts of similar figures.
$\overline{SR} : \overline{PR} = 1:4$	From previous two statements.

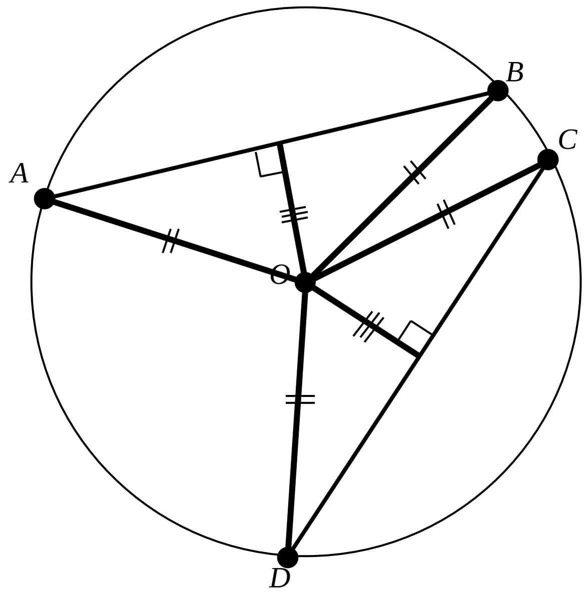
✓✓

(b) Prove that $\overline{SR} : \overline{TS} : \overline{QR} = 2:3:4$

Statement	Explanation
$\angle PST \cong \angle PRQ$	Corresponding angles on parallel lines are equal.
$\Delta PST \cong \Delta PRQ$	AAA
$\overline{TS} = \frac{3}{4}\overline{QR}$	$\frac{\overline{TS}}{\overline{QR}} = \frac{\overline{PS}}{\overline{PR}} = \frac{\overline{PR}-\overline{SR}}{\overline{PR}} = \frac{\overline{PR}-\frac{1}{4}\overline{PR}}{\overline{PR}} = \frac{3}{4}$
$\overline{SR} : \overline{TS} : \overline{QR} = \frac{1}{2}\overline{QR} : \frac{3}{4}\overline{QR} : \overline{QR} = 2:3:4$ As required.	

✓✓

6. [4 marks] *Diagram is not drawn to scale.*



The circle centred at O has chords $\overline{AB}$ and $\overline{CD}$ which are equidistant from O.

(a) Mark the diagram with the shortest lines from O to each chord, such that $\overline{OE}$ is the shortest line from O to $\overline{AB}$ and $\overline{OF}$ is the shortest line from O to $\overline{CD}$. Use appropriate mathematical symbols to record all the known facts of this situation.

✓✓

(b) Prove that chords $\overline{AB}$ and $\overline{CD}$ are congruent by showing in the table below that $\Delta OAB \cong \Delta OCD$
(*May not require the use of every row given in the table.*)

Statement	**Explanation**
$\overline{OE} \cong \overline{OF}$	Given.
$\overline{OB} \cong \overline{OC}$	Radii.
$\angle OEB = \angle OFC = 90°$	Shortest distance from a point to a line is perpendicular to the line.
$\Delta OEB \cong \Delta OFC$	RHS
$\Delta OEA \cong \Delta OFD$	Same explanation as above.
$\Delta OAB \cong \Delta OCD$	As required.

✓✓