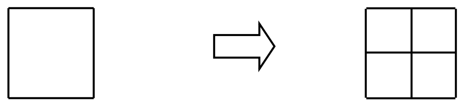
Proof by Induction Assignment

Before you start on this investigation, study proof by induction in *Proof: Interesting Activities in Conjecture and Mathematical Proof*

1. Partitioning a Square into Smaller Squares

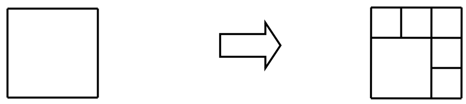
It is easy to partition one square into four smaller squares, or into nine smaller squares.

Diagram 1



The smaller squares do not all have to be the same size as each other, so there are additional possibilities such as:

Diagram 2



A square cannot be partitioned into 2 smaller squares. However, Diagram 1 and Diagram 2 respectively show that 4 squares and 6 squares can be drawn.

Your task is to partition a square into smaller squares, where the smaller squares do not have to be the same size as each other, and find which numbers of smaller squares can be drawn.

(a) Show that a square can be partitioned into 8 squares.

(b) Show that a square can be partitioned into 11 squares.

(c) Explain how Diagram 1 helps to solve the “11 squares” problem once the “8 squares” problem is solved.

(d) Explain how Diagram 1 helps to solve the “14 squares” problem once the “11 squares” problem is solved.

(e) Solve the “7 squares”, “10 squares” and “13 squares” problems.

(f) Prove that a square can be partitioned into any natural number of squares with the exceptions of 2, 3 and 5.

2. Powerful Mathematics, Part One

This challenge is from *Proof: Interesting Activities in Conjecture and Mathematical Proof* :

Do not test this on your calculator: $1 + 3 + 3^2 + 3^3 + \ldots + 3^{99} = \frac{1 - 3^{100}}{1 - 3}$

The equation is correct, but there is a better way to verify that it is correct.

Use proof by induction to show that $1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}$

(a) Use your calculator to evaluate $1 + 3 + 3^2 + 3^3 + 3^4$

(b) Use your calculator to evaluate $\frac{1 - 3^5}{1 - 3}$

What similar fraction has the same value as $1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6$?

(c) Following this pattern, what fraction has the same value as $1 + 5 + 5^2 + 5^3 + 5^4$?

(d) Assume that $1 + x + x^2 + x^3 + \ldots + x^{k-1} = \frac{1 - x^k}{1 - x}$

What similar fraction has the same value as $1 + x + x^2 + x^3 + \ldots + x^{k-1} + x^k$?

(e) What restrictions on the values of *n* and *x* exist for the equation $1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}$?

(f) Construct a proof by induction to show that $1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}$

3. Powerful Mathematics, Part Two

There are equations for adding sequences of numbers all raised to the same power.

For example, $1^2 + 2^2 + 3^2 + 4^2 + 5^2 = \frac{5(5+1)(2\times5+1)}{6}$

For all Natural numbers *n*, the rule is $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(a) Verify that $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true when *n* = 2.

(b) What is the lowest value of *n* for which the equation holds?

(c) Assume that $1^2 + 2^2 + 3^2 + 4^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

What similar fraction has the same value as $1^2 + 2^2 + 3^2 + 4^2 + \dots + k^2 + (k+1)^2$?

(d) Construct a proof by induction to show that $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

4. A False Proof

Required to prove: For all Natural numbers *n*, $\mathbf{1+2+3+\ ...\ +n = \frac{1}{2}\left(n+\frac{1}{2}\right)^2}$

Step One

The equation is correct for $n = 1$

Step Two

Assume the equation is correct for $n = k$, that is $1+2+3+\ ...\ +k = \frac{1}{2}\left(k+\frac{1}{2}\right)^2$

Proof by induction requires that the truth of case $n = k+1$ can be established if case $n = k$ is true,

that is, $1+2+3+\ ...\ +k+(k+1) = \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2$

For case $n = k+1$, L.H.S. = $1+2+3+\ ...\ +k+(k+1)$

$= \frac{1}{2}\left(k+\frac{1}{2}\right)^2 + (k+1)$

$= \frac{1}{2}\left(k^2+k+\frac{1}{4}+2k+2\right)$

$= \frac{1}{2}\left(k^2+2k+1+k+1+\frac{1}{4}\right)$

$= \frac{1}{2}\left((k+1)^2 + 2\times\frac{1}{2}\times(k+1) + \left(\frac{1}{2}\right)^2\right)$

$= \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2$ = R. H. S.

Explain the fault in this false proof.