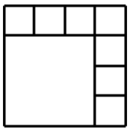
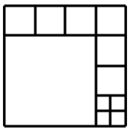
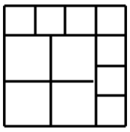
Proof by Induction Assignment Solutions

1. Partitioning a Square into Smaller Squares

(a) 

(b)  or 

(c) and (d)

Diagram 1 shows that any square can be partitioned into four squares, an increase by 3 squares.
Any solution of n squares can be extended to provide a solution of $n + 3$ squares.
If a solution for 11 is available, a solution for 14, 17, 20, … is available.

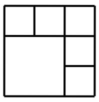
(e) and (f)

Step One

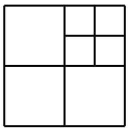
Proof by induction can be applied for higher cases once it is determined that solutions are available for three consecutive cases.

Step Two

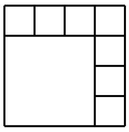
Three consecutive solutions can be found, as illustrated below.



6 squares

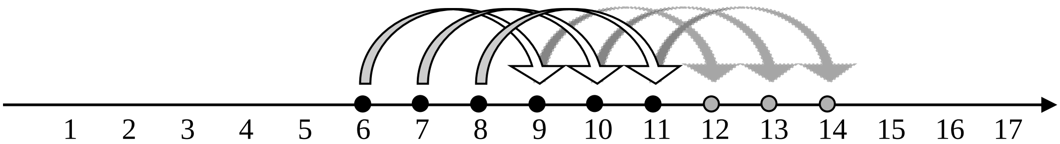


7 squares



8 squares

6, 7 and 8 are possible as shown above, and these are three consecutive cases. Every case can be increased by 3, giving 9, 10 and 11 squares. Then 12, 13 and 14 squares can be drawn, and so on. Therefore every number of squares from 6 is possible.



As 4 is also possible, the only missing numbers of squares are 2, 3 and 5.

2. Powerful Mathematics, Part One

(a) L.H.S. = R.H.S. = 121

(b) The similar fraction is $\frac{1-3^7}{1-3}$

(c) The similar fraction is $\frac{1-5^5}{1-5}$

(d) The similar fraction is $\frac{1-x^{k+1}}{1-x}$

(e) *n* is a Natural number, *x* can take any value except 1 (otherwise the fraction is division by zero).

(f) **Required to prove: for $n \in \mathbb{N}$** $\quad \mathbf{1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}}$

Step One

For $n = 1$, L.H.S. = 1, R.H.S. = $\frac{1-x^1}{1-x}$ = 1 = L.H.S.

Step Two

Assume the equation is correct for $n = k$, that is $1 + x + x^2 + x^3 + \ldots + x^{k-1} = \frac{1-x^k}{1-x}$

For the case $n = k+1$, $1 + x + x^2 + x^3 + \ldots + x^{k-1} + x^k = \frac{1-x^k}{1-x} + x^k$

$$= \frac{1-x^k}{1-x} + x^k = \frac{1-x^k}{1-x} + \frac{x^k(1-x)}{(1-x)} = \frac{1-x^k+x^k-x^{k+1}}{1-x} = \frac{1-x^{k+1}}{1-x}$$

which shows that the rule applies to the $(k + 1)^{th}$ case and completes the proof.

3. Powerful Mathematics, Part Two

(a) L.H.S. = R.H.S. = 5

(b) 1. (Zero is not a Natural number.)

(c) The similar fraction is $\frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

(d) **Required to prove: for $n \in \mathbb{N}$** $\quad \mathbf{1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1-x^n}{1-x}}$

Step One

For $n = 1$, L.H.S. = 1^2 = 1, R.H.S. = $\frac{1(1+1)(2\times1+1)}{6}$ = 1 = L.H.S.

Step Two

Assume the equation is correct for $n = k$, that is $1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

For the case $n = k+1$, $1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)[2k^2+k+6k+6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ which is the formula discovered in part (c) above, therefore the rule applies to the $(k + 1)^{th}$ case and completes the proof.

4. A False Proof

Step One is incorrect. The algebra of Step Two of the false proof is perfect