Proof by Induction Test **Name:**

Formula sheets, one A4 page of notes, the Assignment section and its solutions, and a graphics calculator are permitted. Time = 55 minutes. Total marks available = 30.

1. Party Mix Radio [2,2,4 = 8 marks]

DeeJay programs the continuous party mix on Radio MX every Saturday night. He plays x singles (which run for exactly 3 minutes each) and y extended tracks (exactly 5 minutes each). For example, if Deejay plays 7 singles and 4 extended tracks the party mix will run for exactly 41 minutes.

(a) Find all the values of x and y for which DeeJay's party mix runs exactly 42 minutes.

(b) Obviously, the party mix cannot run for exactly 1 or 2 minutes. Find all other impossible party mix times.

(c) Prove that beyond a certain number of minutes, the number of minutes of the party mix can be any whole number.

2. Powerful cubes [1,2,2,4 = 9 marks]

There are equations for adding sequences of numbers all raised to the same power.

For example, $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = \frac{5^2(5+1)^2}{4}$

(a) Verify the equation above by evaluating both sides.

(b) For all Natural numbers *n* the rule is $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ ... + n^3 = \frac{n^2(n+1)^2}{4}$

Verify that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ ... + n^3 = \frac{n^2(n+1)^2}{4}$ is true when *n* = 2.

What is the lowest value of *n* for which the equation holds?

(c) Assume that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ ... + k^3 = \frac{k^2(k+1)^2}{4}$

What similar fraction has the same value as $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ ... + k^3 + (k+1)^3$?

(d) Construct a proof by induction to show that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ ... + n^3 = \frac{n^2(n+1)^2}{4}$

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please turn over for Question 3

3. Counting With Fingers [1,2,2,1,4 = 10 marks]

A certain mathematics student, let's call him "Fingers", was investigating counting.

Fingers investigated the sum of an arithmetic sequence: $1 + 4 + 7 + 10 + \ ...$ for n terms.

(a) This is the table of his results:

Terms	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Total	1	5	12	22	

Complete the empty cell.

(b) Fingers decided to generalise, and this is what he discovered:

$1 + 4 + 7 + 10 + \cdots$ for n terms $= 1 + (1 + 3 \times 1) + (1 + 3 \times 2) + (1 + 3 \times 3) + \ ... \ + (1 + 3x).$

As there are n terms, find an expression using only n (and not using x) for the final term in Finger's series.

(c) Fingers discovered that he always got the same sum when he added the first term with the final term, the second term with the second-to-last term, the third term with the third-to-last term, and so on. He figured that the sum of the series is given by $\frac{n}{2} \times (2 + 3x)$

Show that Fingers' sum of the series formula is equivalent to $\frac{n(3n-1)}{2}$

(d) For $n = k + 1$ give Fingers' sum of the series formula in terms of k.

(e) Construct a proof by induction to show that the value of $1 + 4 + 7 + 10 + \ ...$ for n terms is given by Fingers' sum of the series formula $\frac{n(3n-1)}{2}$

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4. A False Proof [3 marks]

Required to prove: For all non-negative integers *n*, $\mathbf{2^n = 1}$

Step One

The equation is correct for $n = 0$

Step Two

Assume the equation is correct for all $n \leq k$, that is $2^0 = 1, \quad \dots, \quad 2^{k-1} = 1, \quad 2^k = 1.$

Proof by induction requires that the truth of case $n = k + 1$ can be established if cases $n \leq k$ are true, that is, $2^{k+1} = 1$

For case $n = k + 1$, L.H.S. = 2^{k+1}

$$= \frac{2^{2k}}{2^{k-1}}$$

$$= \frac{2^k \times 2^k}{2^{k-1}}$$

$$= \frac{1 \times 1}{1} = 1 = \text{R.H.S.}$$

Explain the fault in this false proof.