Investigation: Proof by Induction – Solutions and marking key

Question 1. Generous Servings [2, 2, 4 = 8 marks]
Charlie Bucket's Catering aims to provide 50g of Fair Trade chocolate for each guest at a function using a combination of x 200g blocks (4 servings each block)
and y 350g blocks (7 servings each block).

For example, for a function with 55 guests, Charlie might provide 5 200g blocks and 5 350g blocks: $5 \times 4 + 5 \times 7 = 55$ servings, or alternatively he might provide 12 200g blocks and 1 350g block: $12 \times 4 + 1 \times 7 = 55$.

(a) Find all the values of x and y that result in providing 56 servings.

Solutions (*x*,*y*) are (0,8), (7,4) and (14,0)

1 mark	One correct solution
1 mark	All the remaining solutions

(b) Obviously Charlie cannot cater for exactly 1, 2 or 3 servings. Similarly, 5 and 6 servings are impossible. Find all other numbers of servings that are impossible.

Other impossible numbers of servings are 9, 10, 13, 17

1 mark	Identifies 9 and 10 as impossible
1 mark	All the remaining

(c) Prove that beyond a certain number, the number of servings can be any whole number.

1 mark	Identify that if n is possible, n+4 is also possible
1 mark	Conclude that it is sufficient to show four consecutive cases to prove any number from there on.
2 marks	Show that 18, 19, 20 and 21 are all possible and draw conclusion.

Question 2. Inversely Powerful [1, 2, 2, 4 = 9 marks]
There are equations for adding all kinds of sequences. For example:

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}=\frac{3^4-1}{3^4(3-1)}$$

(a) Verify the equation above by evaluating both sides exactly.

1 mark	(a) LHS=$\frac{3^3+3^2+3+1}{3^4}=\frac{40}{81}$; RHS=$\frac{81-1}{81\times 2}=\frac{40}{81}$

(b) For all natural numbers n the rule is

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^n}=\frac{3^n-1}{3^n(3-1)}$$

Verify that this equation is true when $n=2$.

1 mark	LHS=$\frac{1}{3}+\frac{1}{9}=\frac{4}{9}$; RHS=$\frac{9-1}{9\times 2}=\frac{4}{9}$ =LHS

What is the lowest value of n for which the equation holds?

1 mark	$n=1$

(c) Assume that $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}=\frac{3^k-1}{3^k(3-1)}$

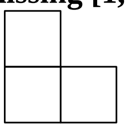
What similar function has the same value as $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}}$?

2 marks	$\frac{3^{k+1}-1}{3^{k+1}(3-1)}$ Give one mark if there are minor errors but some understanding demonstrated.

(d) Construct a proof by induction to show that $\mathbf{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^n}=\frac{3^n-1}{3^n(3-1)}}$

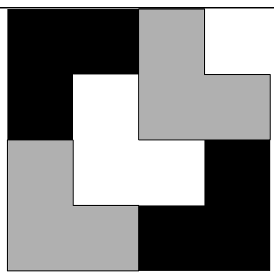
1 mark	Shows that the equation holds for $n=1$ RHS $=\dfrac{3^1-1}{3^1(3-1)}=\dfrac{1}{3}=$ LHS
1 mark	Assume the equation holds for $n=k$, as in (c)
1 mark	For $n=k+1$, correctly state LHS=$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}}=\frac{3^k-1}{3^k(3-1)}+\frac{1}{3^{k+1}}$
1 mark	Correctly shows LHS=RHS as in (c) and draws conclusion

Question 3. Something is missing [1, 2, 2, 1, 4 = 10 marks]

A **right triomino** looks like this: 

A certain mathematician—let's call her Margaret—has cleverly worked out that she can use right triominoes to cover a 4×4 chequerboard with one corner square cut off. Margaret conjectures that she will be able use right triominoes to tile any $2^n\times 2^n$ sized chequerboard with one square cut off.

(a) Show how Margaret might have tiled her 4×4 chequerboard.

1 mark	

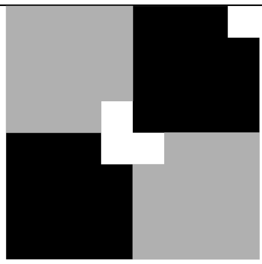
(b) What is the smallest value of n for which Margaret's conjecture is true? Justify your answer (i.e. show that the conjecture is true for that value, and that it can't be true for any smaller value).

1 mark	$n = 1$ giving a 2×2 board
1 mark	A 2×2 board with one square removed *is* a triomino; anything smaller will not be big enough for a single triomino.

(c) Explain why it must be impossible to tile any $2^n \times 2^n$ sized chequerboard with triominoes without removing a square.

1 mark	$2^n \times 2^n$ results in 2^{2n} squares. This cannot be a multiple of 3 squares because all its factors are even. (Other arguments may also be valid.)
1 mark	Clearly set out reasoning.

(d) Show how you could arrange copies of Margaret's 4×4 chequerboard (with extra right triominoes if needed) to make the next size up.

1 mark	

(e) Construct a proof by induction for Margaret's conjecture.

1 mark	Conjecture is true for $n = 1$
1 mark	Assume conjecture is true for $n = k$
1 mark	Show how four copies of the $n = k$ board, plus one triomino can be used to construct the $n = k + 1$ board as in (d)
1 mark	Draws correct conclusion.

Question 4. That's odd: A false proof [3 marks]

Required to prove: $\mathbf{1+3+5+7+\cdots+n=\frac{n(n+1)}{2}}$ **for** $\mathbf{n}>0$

Step One

For $n=1$, R.H.S. = $\frac{1(1+1)}{2}=1$ = L.H.S.

Step Two

Assume the equation is correct for $n=k$, that is, $1+3+5+7+\cdots+k=\frac{k(k+1)}{2}$

Then for $n=k+1$,

$$\text{R.H.S.}=\frac{(k+1)\big((k+1)+1\big)}{2}=\frac{(k+1)(k+2)}{2}$$

$$\begin{aligned}\text{L.H.S.}&=1+3+5+7+\cdots+k+(k+1)\\&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}=\text{R.H.S.}\end{aligned}$$

as required.

Explain the fault in this false proof.

1 mark	k and $k+1$ are not both odd numbers so $\text{L.H.S.}=1+3+5+7+\cdots+k+(k+1)$ is incorrect.
1 mark	The next case after $n=k$ is $n=k+2$
1 mark	Clear explanation