| **Proof by Induction – Validation Test** | |
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| **Total Marks: 30**  **Time Allowed: 60 mins** | |
| **Name:** ______________________________ | **Mark:** |

**INSTRUCTIONS**
You are permitted:

- A calculator
- The "At Home" part of the assignment and its solutions
- Stationery and drawing equipment.

**Question 1. Generous Servings [2, 2, 4 = 8 marks]**
Charlie Bucket's Catering aims to provide 50g of Fair Trade chocolate for each guest at a function using a combination of $x$ 200g blocks (4 servings each block)
and $y$ 350g blocks (7 servings each block).

For example, for a function with 55 guests, Charlie might provide 5 200g blocks and 5 350g blocks: $5 \times 4 + 5 \times 7 = 55$ servings, or alternatively he might provide 12 200g blocks and 1 350g block: $12 \times 4 + 1 \times 7 = 55$.

(a) Find all the values of $x$ and $y$ that result in providing 56 servings.

(b) Obviously Charlie cannot cater for exactly 1, 2 or 3 servings. Similarly, 5 and 6 servings are impossible. Find all other numbers of servings that are impossible.

This Validation Test was prepared by Glen Prideaux, St. Stephen's School, Carramar, Perth, Australia.
*Proof: Interesting activities in conjecture and mathematical proof* is available from *www.pbperth.com*

(c) Prove that beyond a certain number, the number of servings can be any whole number.

**Question 2. Inversely Powerful [1, 2, 2, 4 = 9 marks]**

There are equations for adding all kinds of sequences. For example:

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}=\frac{3^4-1}{3^4(3-1)}$$

(a) Verify the equation above by evaluating both sides exactly.

(b) For all natural numbers $n$ the rule is

$$\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^n}=\frac{3^n-1}{3^n(3-1)}$$

Verify that this equation is true when $n=2$.

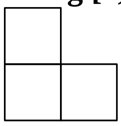
What is the lowest value of $n$ for which the equation holds?

(c) Assume that $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}=\frac{3^k-1}{3^k(3-1)}$

What similar function has the same value as $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^k}+\frac{1}{3^{k+1}}$ ?

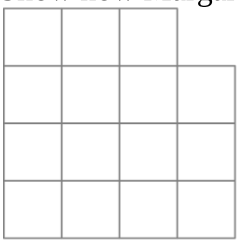
(d) Construct a proof by induction to show that $\mathbf{\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots+\frac{1}{3^n}=\frac{3^n-1}{3^n(3-1)}}$

**Question 3.** **Something is missing [1, 2, 2, 1, 4 = 10 marks]**

A **right triomino** looks like this: 

A certain mathematician—let's call her Margaret—has cleverly worked out that she can use right triominoes to cover a $4 \times 4$ chequerboard with one corner square cut off. Margaret conjectures that she will be able use right triominoes to tile any $2^n \times 2^n$ sized chequerboard with one square cut off.

(a) Show how Margaret might have tiled her $4 \times 4$ chequerboard.



(b) What is the smallest value of $n$ for which Margaret's conjecture is true? Justify your answer (i.e. show that the conjecture is true for that value, and that it can't be true for any smaller value).

(c) Explain why it must be impossible to tile any $2^n \times 2^n$ sized chequerboard with triominoes without removing a square.

(d) Show how you could arrange copies of Margaret's $4 \times 4$ chequerboard (with extra right triominoes if needed) to make the next size up.

(e) Construct a proof by induction for Margaret's conjecture.

**Question 4. That's odd: A false proof [3 marks]**

**Required to prove:** $\mathbf{1+3+5+7+\cdots+n=\frac{n(n+1)}{2}}$ **for** $\boldsymbol{n}>0$

**Step One**

For $n=1$, R.H.S. = $\frac{1(1+1)}{2}=1$ = L.H.S.

**Step Two**

Assume the equation is correct for $n=k$, that is, $1+3+5+7+\cdots+k=\frac{k(k+1)}{2}$

Then for $n=k+1$,

$$\text{R.H.S.}=\frac{(k+1)\big((k+1)+1\big)}{2}=\frac{(k+1)(k+2)}{2}$$

$$\begin{aligned}\text{L.H.S.}&=1+3+5+7+\cdots+k+(k+1)\\&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}=\text{R.H.S.}\end{aligned}$$

as required.

Explain the fault in this false proof.